Exchangeably weighted bootstrap schemes

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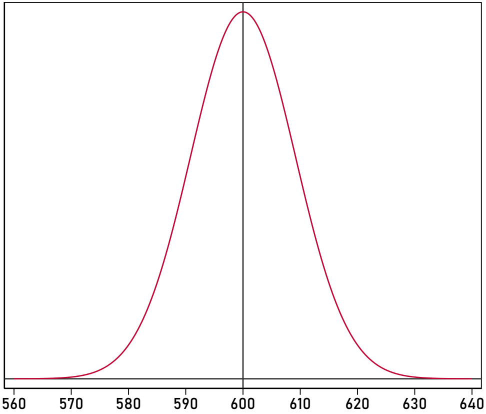
The bootstrap approach to statistical inference



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- How to generate bootstrap samples?
- How to make inference from them (confidence intervals)?

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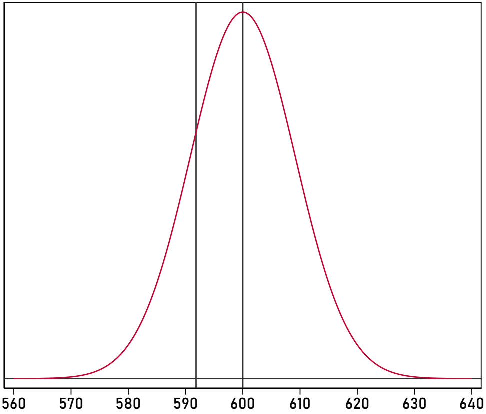

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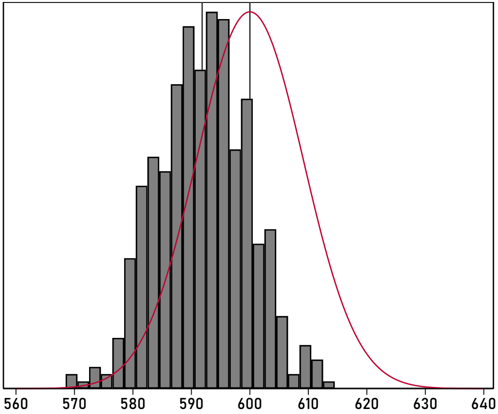

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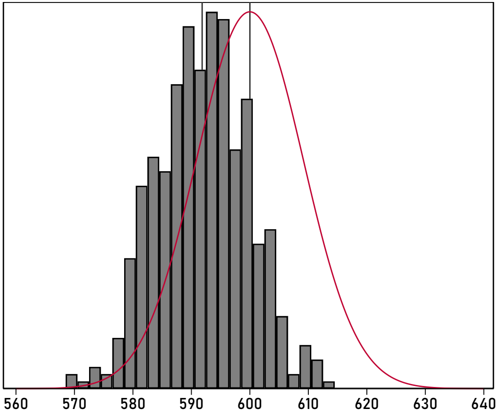

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Generating bootstrap samples

(Efron and Tibshirani, 1993, Davison and Hinkley, 1997)

- Non-parametric bootstrap
  - Classic paired boostraps – `bsample`
  - Block bootstraps – `bsample`
  - Balanced bootstraps – `bsweights` (Kolenikov, 2010)
  - Survey bootstraps – `bsweights`, `rhsbsample` (Van Kerm, 2013);
  - Exchangeable (weighted) bootstrap – `exbsample` (Praestgaard and Wellner, 1993) (also see Chernozhukov et al., 2013)
- Residual bootstrap
  - Wild bootstrap – `boottest` (Roodman et al., 2019)

(Fuzzy classification – incomplete and not mutually exclusive)

Exchangeable (weighted) bootstrap

– Paired bootstrap: obs appear an integer number of times in bootstrap samples

$\Longrightarrow$ 'frequency weighting' of original sample

– Poisson bootstrap: draw from a Poisson(1) distribution to set the bootstrap frequency weight

– Why stick to integer weights? Exponential bootstrap: make a draw from an exponential(1) distribution

  » each observation has a positive (non-integer) weight

  » advantage: no observation is ever 'excluded' from the sample $\Longrightarrow$ no issues of 'no observations' in resamples (e.g., in logits on rare events) or perfect collinearity; bootstrap for matching estimators (Otsu and Rai, 2017))

  » rescale the weights to average to 1 (sum to n) $\Longrightarrow$ Bayesian Bootstrap (Rubin, 1981) (Dirichlet distribution)

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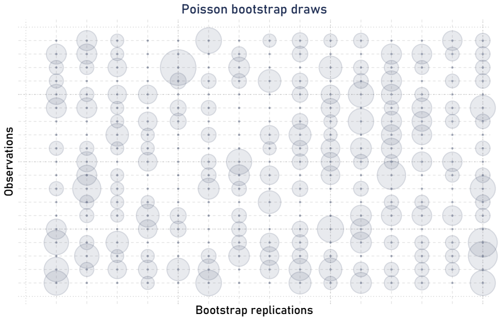
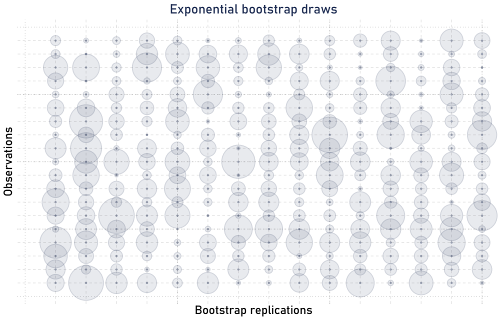
Integer vs non-integer bootstrap weights





A little helper: The `exbsample` command

The command `exbsample` generates bootstrap replication weights using Poisson or Exponential draws

Syntax

`exbsample` [*#*] [*if*] [*in*] [*weight*] [`using` *filename*] [, `stub(`*newvarnameprefix*`)` `distribution(poisson|exponential)` `norescale` `balance(`*#*`)` `strata(`*varlist*`)` `cluster(`*varlist*`)` `frame(`*name*`)` ...]

(A simple command really, but which takes care of nitty-gritty details.)

`ssc install exbsample`

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Using replication weights

- The flexible (but hardest) way: repeat analysis with alternative weight variables
  - e.g., passing weights as argument to do files (and looping):
    `do mydofile.do rweightvar‘i’`
  - `post` results in files (‘resultssets’)

  .. and combine resulting estimates ‘manually’ (allows flexibility in how CIs are constructed)
- Use the `svy bootstrap` prefix (instead of standard `bootstrap:` prefix)
- Use Jeff Pitblado’s `bs4rw` prefix (a predecessor of `svy bootstrap:`)

A simple example

Generate the bootstrap weights

```
. sysuse auto , clear
(1978 Automobile Data)
. exbsample 499 , stub(rw)        // vars rw1 - rw499 created
................................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> .............................
. summarize rw1 rw2 rw3 rw499

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
         rw1 |         74           1    1.014414   .0495382   4.726035
         rw2 |         74           1    1.152799   .0043677   8.042064
         rw3 |         74           1    .9435121   .0204333   3.754344
       rw499 |         74           1     1.12571   .0051524   5.829083
```

Option 1: J Pitblado's bs4rw prefix command

bs4rw (Bootstrapped command needs to accept iweight-s)

```
. qui net install bs4rw , from(http://www.stata.com/users/jpitblado/)
. bs4rw , rweight(rw1-rw499) nodots  : mean price
Mean estimation                   Number of obs    =        74
                                  Replications     =       499

--------------------------------------------------------------
             |  Observed   Bootstrap         Normal-based
             |      Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
       price |  6165.257   328.3822      5521.639    6808.874
--------------------------------------------------------------
. bs4rw mn=r(mean) , rweight(rw1-rw499) nodots  : summarize price
BS4Rweights results                              Number of obs     =        74
                                                 Replications      =       499

      command:  summarize price
           mn:  r(mean)

------------------------------------------------------------------------------
             |  Observed   Bootstrap                         Normal-based
             |     Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          mn |  6165.257   328.3822    18.77   0.000     5521.639    6808.874
------------------------------------------------------------------------------
```

Option 2: `svy bootstrap` prefix

`svy bootstrap` (Bootstrapped command needs to accept iweight-s)

```
. svyset , bsrweight(rw*) vce(bootstrap)
  (output omitted)
. svy bootstrap , nodots : mean price
Survey: Mean estimation                  Number of obs   =         74
                                         Population size =         74
                                         Replications    =        499

--------------------------------------------------------------
             |  Observed   Bootstrap         Normal-based
             |      Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
       price |  6165.257    328.053      5522.285    6808.229
--------------------------------------------------------------

. di el(r(table),2,1)*sqrt(499/498)
328.38223
```

Option 2: `svy bootstrap` prefix

`svy bootstrap` (force non-estimation commands)

```
. svy bootstrap mn=r(mean), nodots force  : summarize price
Bootstrap results                               Number of obs      =        74
                                                Population size    =        74
                                                Replications       =       499

      command:  summarize price
           mn:  r(mean)

------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          mn |   6165.257    328.053    18.79   0.000     5522.285    6808.229
------------------------------------------------------------------------------
```

The benefit of exponential bootstrap in action

```
. bootstrap : logit foreign length i.rep78 if rep78>2
Bootstrap replications (50)
----+--- 1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
................x.....xx...x..x...xx....x..x.....    50
Logistic regression                                  Number of obs =      59
                                                     Replications  =      41
                                                     Wald chi2(3)  =   18.77
                                                     Prob > chi2   =  0.0003
Log likelihood = -19.697108                          Pseudo R2     =  0.4872
  (output omitted)
Note: One or more parameters could not be estimated in 9 bootstrap replicates;
      standard-error estimates include only complete replications.
. bs4rw , rweight(rw1-rw50) : logit foreign length i.rep78 if rep78>2
(running logit on estimation sample)
BS4Rweights replications (50)
----+--- 1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
..................................................    50
Logistic regression                                  Number of obs =      64
                                                     Replications  =      50
                                                     Wald chi2(3)  =   10.44
                                                     Prob > chi2   =  0.0152
Log likelihood = -19.697108                          Pseudo R2     =  0.4872
  (output omitted)
```

Weighted calculations

Generate weighted replication weights

```
. exbsample 499 [iw=weight] , stub(rw) replace // vars rw1 - rw4999 created
................................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> ..............................................................................
> .............................
. bs4rw , rweight(rw1-rw499) nodots : mean price [iw=weight]
Mean estimation                  Number of obs    =         74
                                 Replications     =        499

--------------------------------------------------------------
             |   Observed   Bootstrap          Normal-based
             |       Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
       price |   6568.637   382.1837      5819.571    7317.703
--------------------------------------------------------------
```

Weighted calculations with `svy bootstrap`

Bootstrapped commands must accept *both* iw and pw with `svy bootstrap`

```
. svyset [pw=weight] , bsrweight(rw*) vce(bootstrap)
  (output omitted)
. svy bootstrap , nodots : mean price
Survey: Mean estimation            Number of obs   =         74
                                   Population size =    223,440
                                   Replications    =        499

--------------------------------------------------------------
             |   Observed   Bootstrap         Normal-based
             |       Mean   Std. Err.     [95% Conf. Interval]
-------------+------------------------------------------------
       price |   6568.637    381.8005      5820.322    7316.952
--------------------------------------------------------------
```

Weighted calculations with `svy bootstrap`

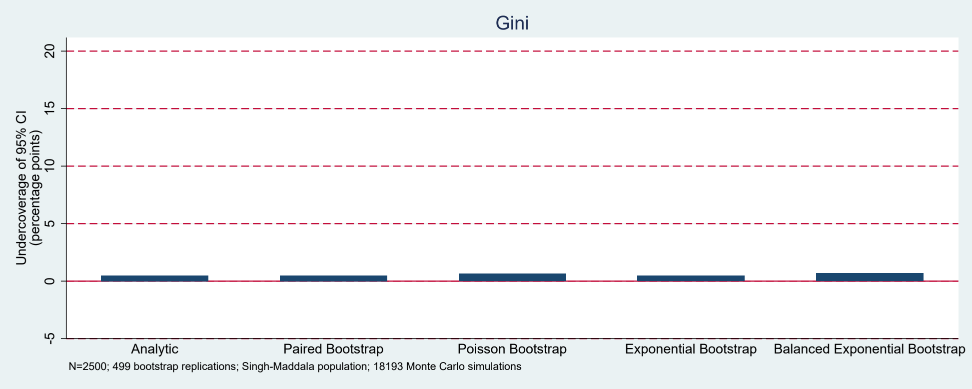
Bootstrapped commands must accept *both* iw and pw with `svy bootstrap`

```
. // convert pw into iw

. pr def mysu ,  properties(svyb)
  1.      if (ustrregexm(`"`0'"' , "\[(\s*pwe?i?g?h?t?\s*=).*\s*\]")==1) {
  2.              loc 0 = subinstr("`0'", "`=ustrregexs(1)'",   "iw=", 1)
  3.      }
  4.      su `0'
  5. end

. svy bootstrap mu=r(mean) , nodots : mysu price

Bootstrap results                               Number of obs     =         74
                                                Population size   =    223,440
                                                Replications      =        499

      command:  mysu price
           mu:  r(mean)

------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          mu |   6568.637   381.8005    17.20   0.000     5820.322    7316.952
------------------------------------------------------------------------------
```

Does it really 'work'?

Statistical properties of exchangeable bootstraps similar to paired bootstrap

Ex.: coverage rate of 95% bootstrapped CI (normal approximation) for inequality measures



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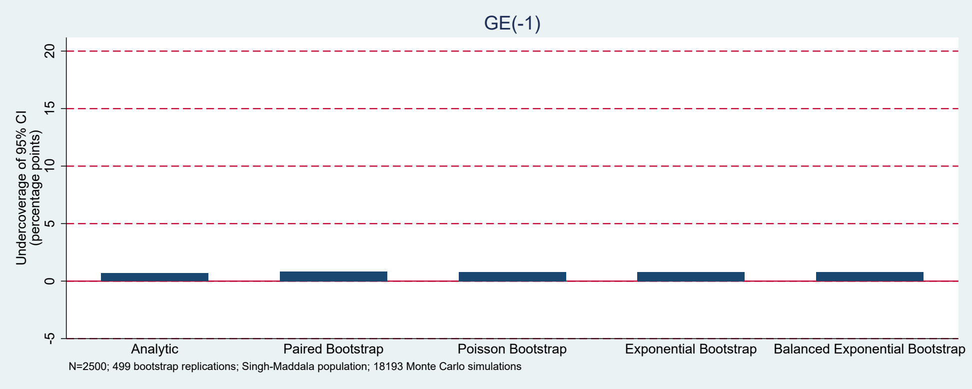

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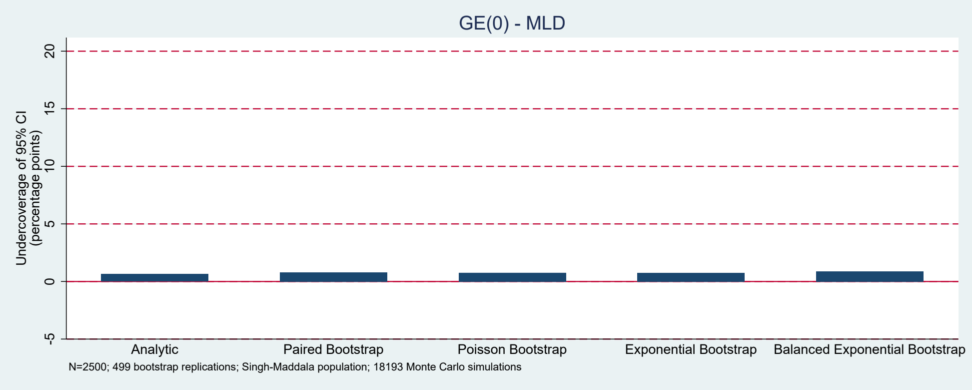

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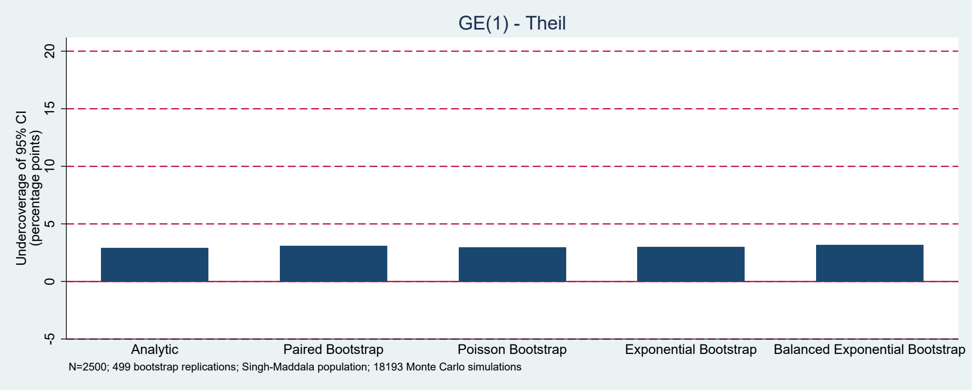

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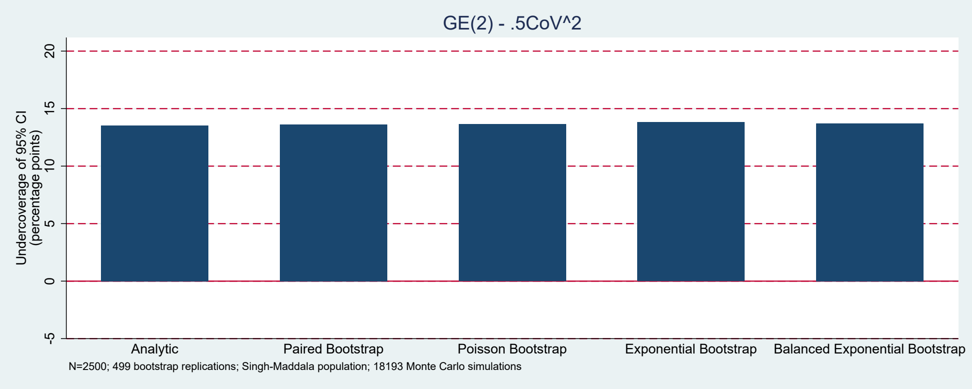

Conclusion

- Exchangeably weighted bootstrap schemes are straightforward and attractive (exponential bootstrap in particular)
- ... and `exbsample` can help
- Exploiting replication weights is admittedly limited if using built-in (prefix) commands only (some further programming for handling replications may be needed for more than small-scale applications)

References

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