scdensity: a program for self-consistent density estimation

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Outline

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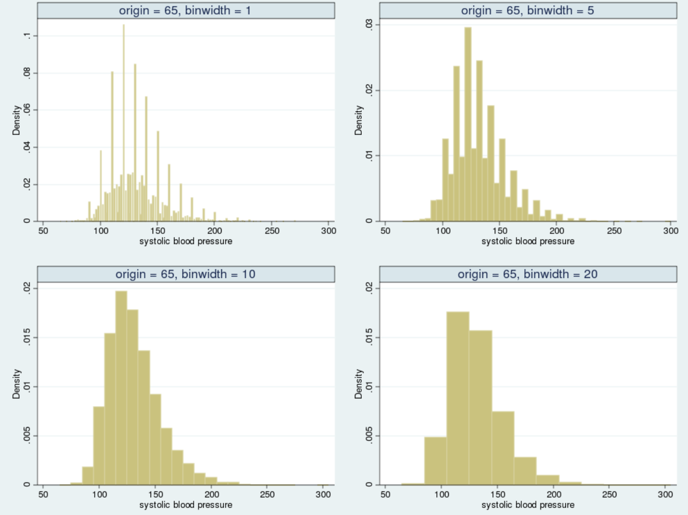
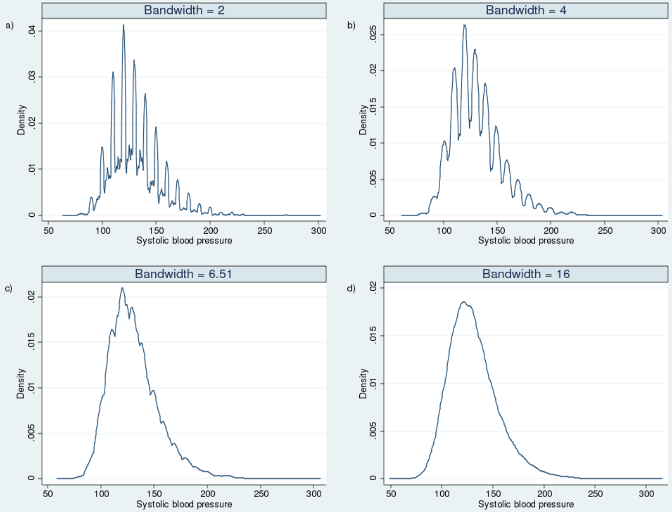
  - Smoothing parameters: trade-off between bias and variance

Histograms with different binwidths



Variable -bpsystol- from dataset nhanes2.dta (-webuse nhanes2-)

Kernel density estimates (Epanechnikov) with different bandwidths



Variable -bpsystol- from dataset nhanes2.dta (-webuse nhanes2-); N=10,351; the bandwidth in graph c) is derived by Stata's default bandwidth rule of thumb

Self-Consistent Density Estimation

- Remember the classical kernel density estimator:

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{N} K\left(\frac{x - X_i}{h}\right) \qquad (1)$$

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- The basic idea of the self-consistent method is *not* to search for an optimal bandwidth, given an arbitrary kernel function...
- ...but to find an optimal shape of the kernel, given the data.
- No parameters need to be fixed beforehand.

scdensity: the program

Syntax

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- `scdensity` is available from SSC: `ssc install scdensity`
- `help scdensity` for further information

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  - Three fixed bandwith rules of thumb:
    1. $h_o = 0.9\min(\sigma, IQ/1.349)n^{-(\frac{1}{5})}$
    2. $h_o = 1.06\min(\sigma, IQ/1.349)n^{-(\frac{1}{5})}$
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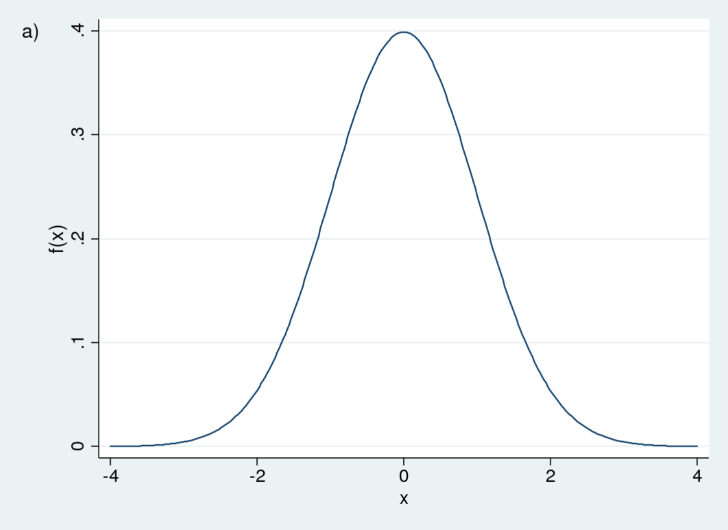
  - See [Silverman, 1998], [Haerdle et al., 2004], [Scott, 1992], respectively.
  - Variable bandwidth estimation (aka adaptive kernel).
  - The user written -kdens- (available from SSC [Jann, 2005], [Jann, 2007]) was used for kernel density estimation.
  - The user written -fmm- (SSC, [Deb, 2007]) was used for fitting maximum likelihood mixture models.

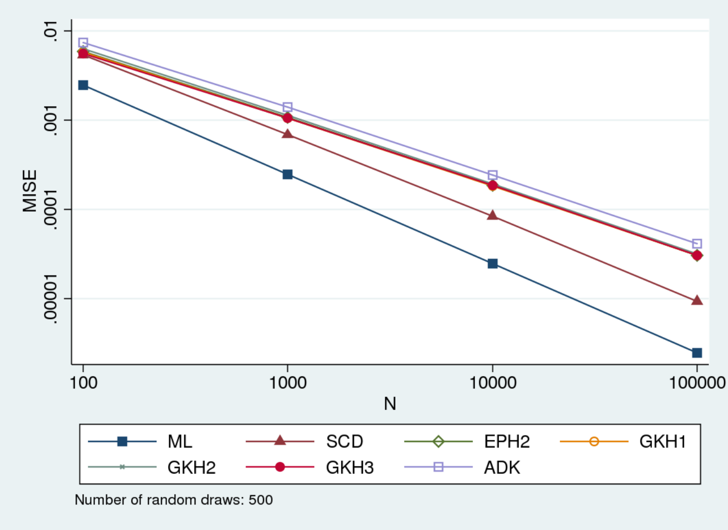
Results

- Abbreviations:
- ML = maximum likelihood
- SCD = self-consistent method
- EPH2 = Epanechnikov kernel with bandwidth #2 from previous slide
- GKH1 = Gaussian kernel with bandwidth #1 from previous slide
- GKH2 = Gaussian kernel with bandwidth #2 from previous slide
- GKH3 = Gaussian kernel with bandwidth #3 from previous slide
- ADK = adaptive kernel (Epanechnikov)

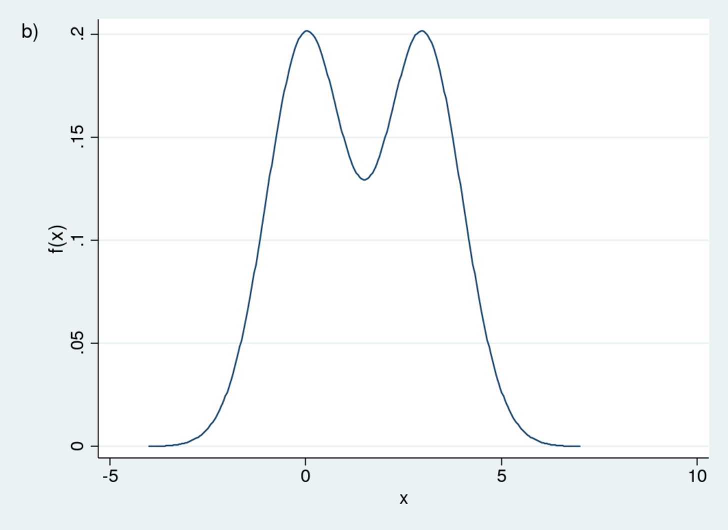
Test density a):
$\phi(\mu,\sigma^2) = (2\pi)^{-\frac{1}{2}}\sigma^{-1} exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\}$



Results for test density a)



Test density b): $f(x) = \frac{1}{2}\phi(0,1) + \frac{1}{2}\phi(3,1)$



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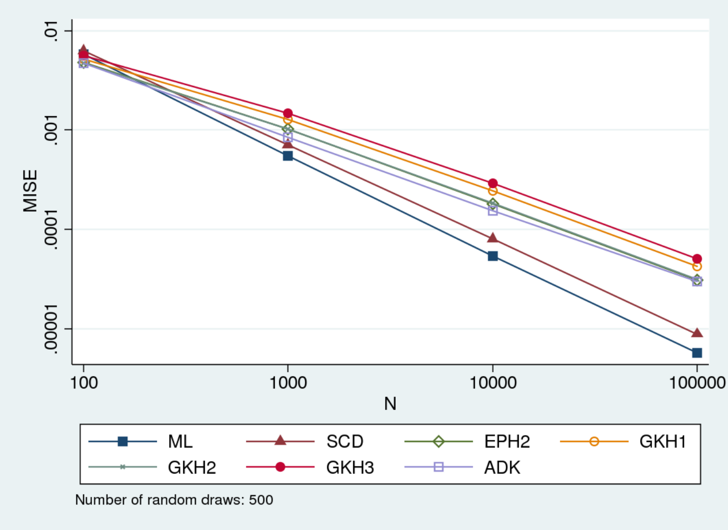
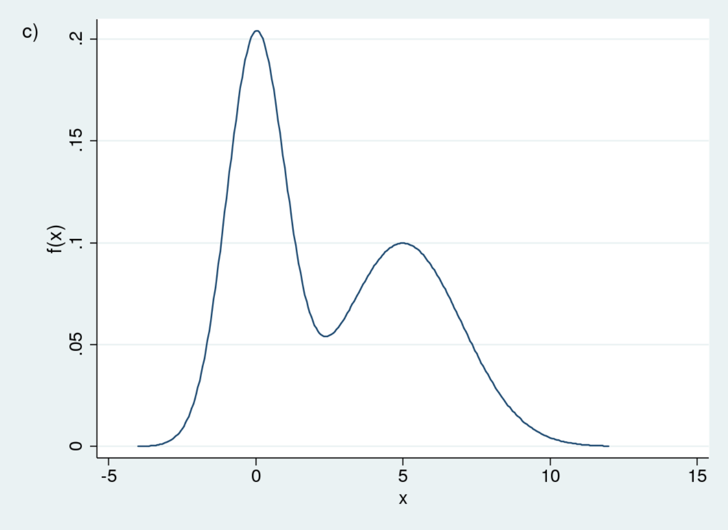


Test density c): $f(x) = \frac{1}{2}\phi(0,1) + \frac{1}{2}\phi(5,2^2)$



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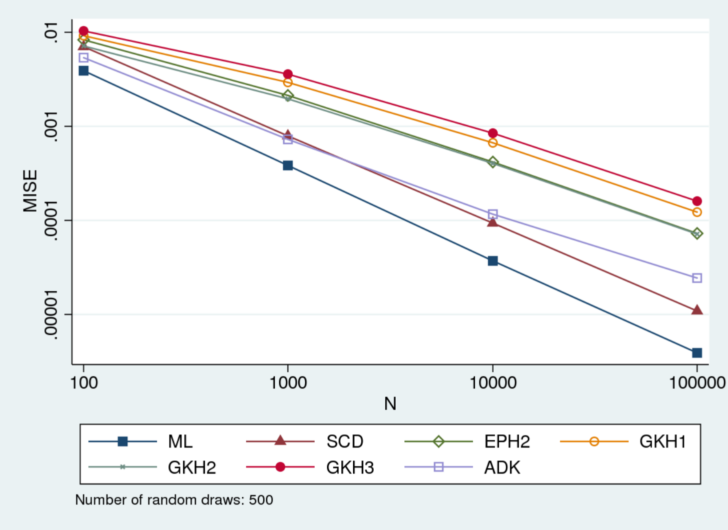
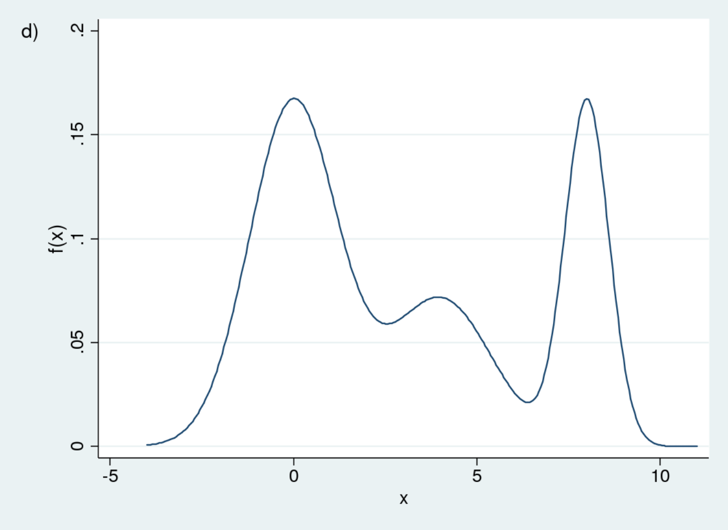


Test density d):

$$f(x) = \frac{1}{2}\phi(0, 1.2^2) + \frac{1}{4}\phi(4, 1.4^2) + \frac{1}{4}\phi(8, 0.6^2)$$



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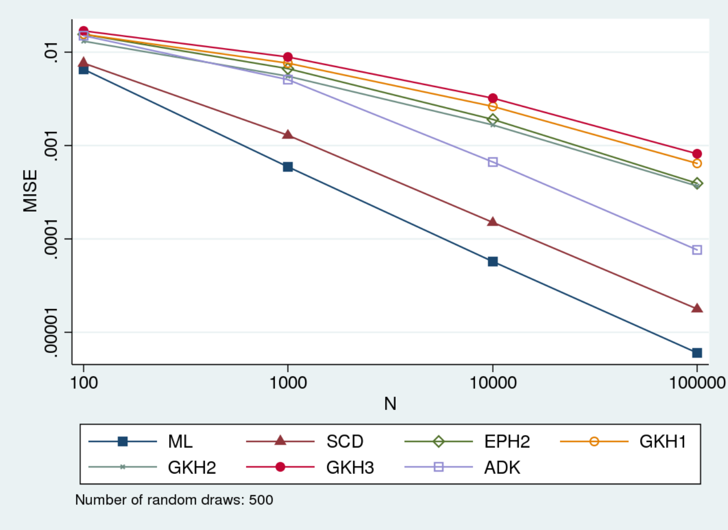

Conclusion

- Given the test densities and kernel density estimators used in the simulations, the self-consistent method was the most accurate among the nonparametric estimators.

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- For one of the test densities ($f(x) = \frac{1}{2}\phi(0,1) + \frac{1}{2}\phi(3,1)$)
- ...the self-consistent method performed nearly as well as the (parametric) ML estimate
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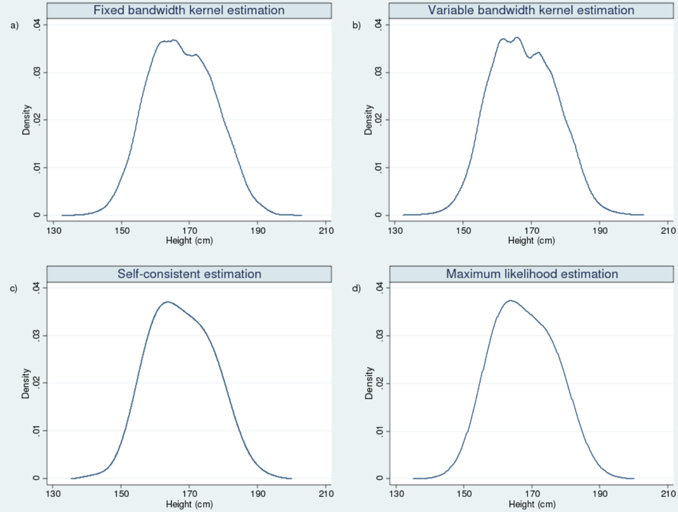
- Given the test densities and kernel density estimators used in the simulations, the self-consistent method was the most accurate among the nonparametric estimators.
- For one of the test densities $(f(x) = \frac{1}{2}\phi(0,1) + \frac{1}{2}\phi(3,1))$
- ...the self-consistent method performed nearly as well as the (parametric) ML estimate
- ...without relying on any prior assumptions or parameter fixations.
- The question remains: Is it of practical importance?
- Yes, it certainly can be of practical importance. The following figure shows an example:

Comparison of density estimates using real data



Variable -height- from the dataset nhanes2.dta (-webuse nhanes2-); N=10,351; graph a): Epanechnikov kernel with bandwidth rule #1, Stata's default.

Outlook

- Program features
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  - Weights
  - Grid expansion

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  - Weights
  - Grid expansion
- Non- and semiparametric models
  - Bivariate density estimation
  - Smoothing & regression

References (1)

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Annals of Mathematical Statistics **34** (2), 480–491.

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Thank you!

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Outline of the basic algorithm of the self-consistent estimator (1)

- Departure: an optimal convolution kernel can be derived for known densities [Watson & Leadbetter, 1963]
- The Fourier transform $K_{opt}(t)$ of the optimal kernel $K_{opt}(x)$ equals

$$K_{opt}(t) = \frac{N}{N - 1 + |\omega(t)|^{-2}} \tag{3}$$

- where $\omega(t)$ is the Fourier transform of the true density $f(x)$
- Then, the Fourier transform of the density estimate in equation (2) is

$$\hat{\omega}(t) = \Delta(t) K_{opt}(t) = \frac{N\Delta(t)}{N - 1 + |\omega(t)|^{-2}} \tag{4}$$

Outline of the basic algorithm of the self-consistent estimator (2)

- ...where $\Delta(t)$ is the emprical characteristic function

$$\Delta(t) = \frac{1}{N}\sum_{i=1}^{N} exp(itX_i) \tag{5}$$

- $K_{opt}(t)$ is of course only known if the true density is known.
- The self-consistent method now uses equation (4) for which the unknown term ω is replaced with an initial guess $\hat{\omega}_0$,
- ...which results in the estimate $\hat{\omega}_1$.
- Then the improved estimate $\hat{\omega}_2$ is obtained by using a kernel which is optimal for $\hat{\omega}_1$, and so on.

Outline of the basic algorithm of the self-consistent estimator (3)

- This is iterated until a certain point in the sequence

$$\hat{\omega}_{n+1} = \frac{N\Delta}{N - 1 + |\hat{\omega}_n|^{-2}} \tag{6}$$

- ...is reached, for which

$$\hat{\omega}_{sc} = \frac{N\Delta}{N - 1 + |\hat{\omega}_{sc}|^{-2}} \tag{7}$$

- See [Bernacchia & Pigolotti, 2011] for a detailed description.